Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 46 (2025), No. 5, pp. 1385–1400
DOI : 10.47974/JIOS-1338

Large deviations for spatial telecommunication systems : The Boolean model

Andrew K. Boahen †
African Institute of Mathematical Sciences
Kasa Global Villa – 1
Shoppers Street Manet
Spintex Accra
Ghana

Thomas Katsekpor *
Department of Mathematics
College of Basic and Applied Sciences
University of Ghana
P. O. Box LG 62 Accra
Ghana

Kwabena Doku-Amponsah §
Department of Statistics and Actuarial Science
College of Basic and Applied Sciences
University of Ghana
Box LG 115 Accra
Ghana

Abstract

There have been some concerns that telecommunication companies are facing today which are spatial in nature. Systems operated by these telecommunication companies have evolved along the years. The main concern of these companies is the ability to provide quality service to customers or users in a dense regime. Therefore, they sought to answer questions such as (i) what is the best possible configurations of base stations and users that maximizes quality service? (ii) can one estimate and control the probability of bad service, which may be seen as a rare event? and many more which may arise during companies operations. To answer these questions one will often have to estimate the tail distribution of events, which may be considered under the realm of large deviations. In this article, we define the empirical

† *E-mail:* `aboahen@aims.edu.gh`
* *E-mail:* `tkatsekpor@ug.edu.gh (Corresponding Author)`
§ *E-mail:* `kdoku-amponsah@ug.edu.gh`

marked measure of the Boolean model, which will serve estimate the intensity measure of the Marked Poisson Point Process of devices or users, and the empirical connectivity measure of the Boolean Model which will serve estimate the coverage probability density of the spatial telecommunication area. For these empirical measures, we establish large deviation principle (LDP) in the τ – topology.

Subject Classification: *60F10, 05C80, 68Q87.*

Keywords: *Large deviations, Empirical measures, Relative entropy, Boolean model, Marked poisson point processes, Coverage probability, Poisson rate.*

1. Introduction

The number of subscribers of Telecommunication networks have witnessed a major growth in the last twenty years due to the easy accessibility provided by continuous improvements in telecommunication devices. The growth in number of subscribers mostly create artificial disturbances in the telecommunication system, which may attributed to bad connection of devices and/or bad service. The topology of telecommunication systems give mathematicians a very rich structure for research or further study. The mathematics of telecommunication networks has been become popular among significant number of researchers. Researchers have noted that the spatial arrangement of devices in the telecommunication area is key aspect of the network. See, Baccelli and Blaszczyszyn [1],[2]. Stochastic geometry is the main tool developed and used by researchers interested in mathematical modelling of telecommunication systems. See [1], [2], [7].

The main concern of these researchers are the issues of connectivity coverage area and connectivity in the modelling of telecommunication networks. The topic of coverage area involves knowing how far the signal originating from communication devices can spread across the interaction area and connectivity which solves the problem of how far a message can travel in a given messaging area. Therefore, manipulation of the connectivity and the coverage area of a telecommunication system to achieve a desire results may be seen as problem of reversing a deteriorating services quality. The manipulations usually involve control and/or estimation of the probability of occurrence by telecommunication engineers. See Hirsch et al. [8], and the mathematics used to accomplish such called large deviations.

The Large deviations theory has been used to find asymptotic equipartition properties for hierarchical modelled as multitype Galton-

Watson tress and networked structures modelled as coloured random graphs. See, Doku-Amponsah [5]. The main technique is to code the properties of the graphs (including tress) in suitably defined empirical distributions. Then using the method of exponential change of measure, the method of types and the method of mixtures the (joint) large deviation principle of the empirical measures was established.

A large deviation principle for the empirical measure of connectable receivers in a wireless network modelled as the SINR graph was proved by [8]. The main tool used in the article is the contraction principle and Dawson-Gärtner technique. See, Dembo and Zeitouni [6, Theorem 4.6.1].

Umar etal. in [13] had established a joint LDP of the empirical spin measure and the empirical bond measure of the uniformly random d-regular graph. The authors of this article gave spin values to the nodes of the graph via the Potts model and by the method of types the authors derived the rate function of the joint LDP. See, [13]. In Sakyi-Yeboah et al.[11] the authors proved large deviations results of the super-critical telecommunication networks modelled as a Signal-to-Interference-Noise-Ratio (SINR) network. In this article, we derived the joint large deviation principle for the empirical marked distribution and empirical pair distribution in a telecommunication network modelled via the Boolean random network.

The remaining aspect of this paper are organized as follows: In Section 2 we present the boolean random network (BRN) and the methods used to prove the LDPs of the proposed empirical distributions of the BRN. In Section 1 we compute the connectivity probability between two users, and impose some important assumptions for convergence of this probability. We present in Section 2 the main results of the paper. In Section 3 we present the LDP for the empirical marked measure and the proof. Our Section 4 presents conditional LDP for the empirical pair measure given empirical mark measure. Finally, our Section 4 gives the conclusions of the study.

2. Statement of main results

2.1 *Boolean Modelling of Telecommunication Systems*

Let $d \in \mathbb{N}$ be fix number called the dimension and $\Gamma \subset \mathbb{R}^d$ be a set which is measurable taken against $B(\mathbb{R}^d)$, the borel-sigma field. By *Leb* : $\mathbb{R}^d \to [0,\infty]$ we denote the lebesgue measure and by $\nu_\lambda : \Gamma \to [0,1]$, while $\nu_\lambda(\Gamma) = 1$ a Poisson rate (rate). For this $\nu_\lambda : \Gamma \to$ $[0,1]$ and $P_\lambda : \Gamma \to \mathbb{R}_+$

serving as coverage probability measure, the boolean network may be defined as follows:

- Pick a Poisson Point Process (PPP), $\mathbb{Y} = (Y_i)_{i \in I}$ according to an intensity measure ν_λ denoting the structure of devices or users in $\Gamma \subseteq \mathbb{R}^d$, the device interaction space.
- For $\mathbb{Y}$, pick assign each Y_i a random coverage area $D_{Y_i} = D_i$ independently agreeing with P_λ the law of the volume of the balls contained in Γ.
- Given any two points $[(Y_i, D_i), (Y_j, D_j)]$ we connect an edge iff $D_i \cap D_j \neq \varnothing$.

We consider $Y(\mathbb{Y}, P_\lambda) = \{[(Y_i, D_i), i \in \ell], E\}$ with respect to the combined law of the marked process, see [12], and the random network. We shall call $Y_i := (Y_i, D_i)$ the interaction area of user i and Y a boolean random network(BRN). We assume that there is a sequence $b_\lambda \in \mathbb{R}$, and P, ν some functions with the property that $\lambda b_\lambda \to 1$ or $\lambda b_\lambda \to 0$ or $\lambda b_\lambda \to \infty$ implies

$$b_\lambda P_\lambda(t \mid x)\nu_\lambda(x-y)P_\lambda(t \mid y) \to P(t \mid x)\nu(x-y)P(t \mid y).$$

We shall limit this study to the case where $\lambda b_\lambda \to 1$. Thus, our main focus in this article is the study of the LD analysis of the behaviour of the critical Bolean random networkd. Now, write $\mathcal{D} := \{D(y,h) \subseteq \Gamma : h \in \mathbb{R}_+, y \in \Gamma\}$ and $\mathcal{Y} := \cup_{a \in \mathcal{D}} \{a\}$. Denote by $\mathcal{N}(\mathcal{Y} \times \mathcal{D})$ the set of all measures(positive) on $\mathcal{Y} \times \mathcal{D}$ endowed with the $\tau-$ topology. Hereafter, we shall call $\mathcal{Y}$ locally finite subset of the set Γ. For each BRN Y we associate a probability distribution, the empirical mark measure, $L_1 \in \mathcal{N}(\mathcal{Y} \times \mathcal{D})$, by

$$L_1([y,a]) = \frac{1}{\lambda} \sum_{i \in \ell} \delta_{Y_i}(y,a)$$

and a finite measure (Symmetric), the empirical connectivity distribution $L_2 \in \mathcal{N}(\mathcal{Y} \times \mathcal{D} \times \mathcal{Y} \times \mathcal{D})$, by

$$L_2(x, b_1, y, b_2) := \frac{1}{\lambda} \sum_{(i,j) \in E} [\delta_{(Y_i, Y_j)} + \delta_{(Y_j, Y_i)}](x, b_1, y, b_2).$$

We observe that the empirical marked measure is a probability distribution and the absolute value of the empirical connectivity distribution is $2|E|/\lambda^2$. Moreover, $\| L_1 \|$ is the number of users communicating with device i. Let each ball be very small and contain a finite number of devices. Further, we assume that the location of a user does not intersect the location of other users.

Proposition 2.1: (Connectivity Distribution). Suppose Y is an BRN with rate $\nu_\lambda : \Gamma \to \mathbb{R}_+$ and P_λ is the probability distribution of the volume of the balls centered on the points of the $\mathbb{Y}$. Then the probability that two users at positions $Y_i = y$ and $Y_j = z,\ j \neq i$ are connected is given by

$$q_\lambda(y, b_y, z, b_z) := [1 - e^{-\nu_\lambda(b_y - b_z)P_\lambda(b_y)P_\lambda(b_z)}],\ (y, b_y, z, b_z) \in \mathcal{Y} \times \mathcal{D} \times \mathcal{Y} \times \mathcal{D},$$

where, $b_y = D(y, r_y),\ b_z = D(z, r_z),$ and $r_y, r_z \in (0, \infty)$ $b_y - b_z = \{x_1 - x_2 : x_1 \in b_y, x_2 \in b_z\}$.

Proof: Suppose $Y_j = x$ and $Y_i = y$ are the positions of two devices in an interaction space Γ. Note that any two devices will communicate when their coverage spaces overlap each other. Then we have,

$$\begin{aligned} q_\lambda(Y_i, D_i, Y_j, D_j) &= \mathbb{P}(D(Y_i, r_i) \cap D(Y_j, r_j) \neq \varnothing) \\ &= 1 - \mathbb{P}(D(Y_i, r_i) \cap D(Y_j, r_j) = \varnothing) \\ &= 1 - e^{-\mathbb{E}[\nu_\lambda(D_i - D_j)]} \end{aligned} \tag{2.1}$$

Now, we observe that

$$\begin{aligned} \mathbb{E}[\nu_\lambda(D_i - D_j)] &= \mathbb{E}[\mathbb{E}[\nu_\lambda(D_i - D_j) \mid D_j]] \\ &= \mathbb{E}\left[\int \mathbb{1}_{D_i(b_2)} \nu_\lambda(b_2 - D_j) P_\lambda(db_2)\right] \\ &= \int \mathbb{1}_{D_j(b_1)} \int \mathbb{1}_{D_i(b_2)} \nu_\lambda(b_2 - b_1) P_\lambda(db_2) P_\lambda(db_1) \\ &= \iint \mathbb{1}_{D_j(b_1)}\, \mathbb{1}_{D_i(b_2)} \nu_\lambda(b_2 - b_1) P_\lambda(db_2) P_\lambda(db_1) \\ &= P_\lambda(D_i) \nu_\lambda(D_i - D_j) P_\lambda(D_j) \end{aligned} \tag{2.2}$$

Hence , probability of two devices been connected is given by,

$$q_\lambda(Y_i, D_i, Y_j, D_j) = 1 - e^{-P_\lambda(D_i)\nu_\lambda(D_i - D_j)P_\lambda(D_j)}.$$

We hereafter assume that as $\lambda \to \infty$,

$$\lambda^2 P_\lambda \to P \quad \text{and} \quad \lambda^{-3} \nu_\lambda \to \nu.$$

We note that $\nu_\lambda P_\lambda P_\lambda \approx P\nu P / \lambda$, and therefore, using the Taylor expansion around the origin 0 we have the expression;

$$\begin{aligned} q_\lambda(Y_i, D_i, Y_j, D_j) &= 1 - [1 - P_\lambda(D_i)\nu_\lambda(D_i - D_j)P_\lambda(D_j) + O(\lambda^{-2})] \\ &= \nu_\lambda(D_i - D_j)P_\lambda(D_i)P_\lambda(D_j) + O(\lambda^{-2}). \end{aligned}$$

Now, we multiple through by $\lambda,$ and we take limit as $\lambda \to \infty$ to obtain

$$\lambda q_\lambda(Y_i, D_i, Y_j, D_j) \to P(D_i)\nu(D_i - D_j)P(D_j) := \Psi(D_i, D_j),$$

which proves the proposition.

Proposition 2.2: Suppose $G_1, G_2, \cdots, G_n$ is a decomposition of $\mathcal{Y} \times \mathcal{D}$. Then, for $n < \lambda,$ the probability Law of the random distribution L_1 satisfies the inequality:

$$\sum_{i=1}^{n} \log\left[\frac{e^{-\nu_\lambda \otimes P_\lambda(G_i) - \varepsilon}(\nu_\lambda \otimes P_\lambda(G_i) - \varepsilon)^{\lambda\omega(G_i)}}{(\lambda\omega(G_i))!}\right] \le \log P(L^1 = \omega)$$
$$\le \sum_{i=1}^{n} \log\left[\frac{e^{-\nu_\lambda \otimes P_\lambda(G_i) + \varepsilon}(\nu_\lambda \otimes P_\lambda(G_i) + \varepsilon)^{\lambda\omega(G_i)}}{(\lambda\omega(G_i))!}\right] + \eta_n, \qquad (2.3)$$

where, $\lim_{n\to\infty}\lim_{\lambda\to\infty}[\eta_n(\lambda, G_1, \cdots, G_n))/\lambda] = 0$ and $\nu_\lambda \otimes P_\lambda(x, b_x) = \nu_\lambda(x)P_\lambda(b_x)$ is the product distribution.

2.2 *Results*

Theorem 2.3: *Suppose Y is a BRN with rate $\nu_\lambda : \Gamma \subseteq \mathbb{R}^d \to \mathbb{R}_+$ and P_λ is the coverage law of Y. Let the rate $\nu_\lambda : \Gamma \to \mathbb{R}_+$ satisfy $\lambda^{-3}\nu_\lambda \to \nu$ and $P_\lambda : \mathcal{D} \to (0,1)$ satisfy $\lambda^2 P_\lambda \to P$. Then, as $\lambda \to \infty$, the pair (L_1, L_2) obeys an LDP in the space $\mathcal{N}(\mathcal{Y} \times \mathcal{D}) \times \mathcal{N}(\mathcal{Y} \times \mathcal{D} \times \mathcal{Y} \times \mathcal{D})$ on the scale λ and rate function*

$$I(\sigma,\mu) = \begin{cases} H(\sigma \| \nu \otimes P) + \frac{1}{2}[H(\mu \| \psi\, \sigma \otimes \sigma) + \|\psi\, \sigma \otimes \sigma\| - \|\mu\|] & \text{if } \|\mu\| < \infty, \\ \infty, & \text{otherwise.} \end{cases} \qquad (2.4)$$

Corollary 2.3: Suppose Y is a BRN with Poisson rate $\nu_\lambda : \Gamma \to \mathbb{R}_+$ and P_λ is the coverage law of Y. Let the rate be $\nu_\lambda(dx) = \lambda^3 dx$ and $P_\lambda(db_x) = \frac{4}{3}\pi r_x^3 / \lambda^2 Vol(D)$ be the coverage law. Then, as $\lambda \to \infty$, (L_1, L_2) obeys an LDP in the space $\mathcal{N}(\mathcal{Y} \times \mathcal{D}) \times \mathcal{N}(\mathcal{Y} \times \mathcal{D} \times \mathcal{Y} \times \mathcal{D})$ on the scale λ and convex rate function

$$I(\sigma,\mu) = \begin{cases} H(\sigma \| \nu \otimes P) + \frac{1}{2}[H(\mu \| \psi\, \sigma \otimes \sigma) + \|\psi\, \sigma \otimes \sigma\| - \|\mu\|] & \text{if } \|\mu\| < \infty, \\ \infty, & \text{otherwise.} \end{cases} \qquad (2.5)$$

where

$$\Psi(a_x, a_y) = \frac{16}{9}\mu^2 r_x^3 r_y^3 \left[\frac{Vol(a_x - a_y)}{Vol(\Gamma)^2}\right],$$

and $Vol(D)$ is the volume of the space D

Therefore, from the corollary, one can infer that the average number of connectivity per users converge is given by

$$\| E \| / \lambda \to \frac{8\mu^2}{9Vol(D)^2} \int_{D\times\mathcal{D}} \int_{D\times\mathcal{D}} r_x^3 r_y^3 Vol(b_x - b_y) P(dbx) P(dby) dxdy \text{ in probability.}$$

3. LDP for L_1

Using the method of types we proof an LDP for $L_1 = \sigma$ as $\lambda \to \infty$ in this Section. To begin we state the LDP for $L_1 = \sigma_\lambda$ below:

Lemma 3.1: *Suppose* Y *is a BRN with rate* $\nu_\lambda : \Gamma \subseteq \mathbb{R}^d \to \mathbb{R}_+$ *and* P_λ *is the coverage probability distribution of* Y. *Let the rate* $\nu_\lambda : \Gamma \to \mathbb{R}_+$ *satisfy* $\lambda^{-3}\nu_\lambda \to \nu$ *and* $P_\lambda : \mathcal{D} \to (0,1)$ *satisfy* $\lambda^2 P_\lambda \to P$. *Then,*

$$e^{-\lambda H(\sigma^{(n)} \| \nu^{(n)} \otimes P^{(n)}) + \gamma_1(\lambda)} \le \mathbb{P}(L_1 = \sigma) \le e^{-\lambda H(\sigma^{(n)} \| \nu^{(n)} \otimes P^{(n)}) + \gamma_2(\lambda)} \qquad (3.)$$

$$\lim_{\lambda\to\infty} \gamma_1(\lambda) = 0,$$

$$\lim_{\lambda\to\infty} \gamma_2(\lambda) = \lim_{\lambda\to\infty} \frac{1}{\lambda} \gamma_n(\lambda, G_1, \ldots, G_n),$$

while $\sigma^{(n)}$ and $\nu^{(n)} \otimes P^{(n)}$ are the coarsening projections of σ and $\nu \otimes Q$ on the decomposition $(G_1, \ldots, G_n)$.

The proof for Lemma 3.1 is based the Stirling's formula below:

Proof: Choose λ large enough and observe that the upper bound of the Equation (2.3) is given by

$$\log \mathbb{P}(L_1 = \sigma) \le \sum_{i=1}^{n} \{-\lambda\nu \otimes P(G_i) + \lambda\sigma(G_i) \log[\lambda\nu \otimes P(G_i)] - \log[\lambda\sigma(G_i)]!\} + \sigma_n(\lambda, G_1, \ldots, G_n)$$

Using the the Stirling's formula, we have

$$\log \mathbb{P}(L_1 = \sigma) \le \sum_{i=1}^{n} \{-\lambda\nu \otimes P(G_i) - \log[(2\mu)^{\frac{1}{2}} (\lambda\sigma(G_i))^{\lambda\sigma(G_i) + \frac{1}{2}} e^{-\lambda\sigma(G_i)}]\} + \sum_{j=1}^{n} \left\{ \frac{1}{12\lambda\sigma(G_j)+1} + \lambda\sigma(G_j) \log[\lambda\nu \otimes Q(A_j)] \right\} + \gamma_n(\lambda, G_1 \ldots, G_n)$$

$$\log \mathbb{P}(L_1=\sigma) \le \sum_{j=1}^{n}\left\{-\lambda\nu\otimes P(G_j)-\frac{1}{2}\log(2\mu)-\left[\lambda\sigma(G_j)+\frac{1}{2}\right]\log[\lambda\sigma(G_j)]+\lambda\sigma(G_j)\right\}$$
$$+\sum_{j=1}^{n}\left\{\frac{1}{12\lambda\sigma(G_j)+1}+\lambda\sigma(G_j)\log[\lambda\nu\otimes P(G_j)]\right\}+\gamma_n(\lambda,G_1\ldots,G_n)$$
$$\log \mathbb{P}(L_1=\sigma) \le \sum_{i=1}^{n}\left\{-\lambda[\nu\otimes P(G_j)-\sigma(G_j)]-\lambda\sigma(G_j)\log\frac{\sigma(G_j)}{\nu\otimes P(G_j)}-\frac{1}{2}\log[\lambda\sigma(G_j)]\right\}$$
$$+\sum_{i=1}^{n}\left\{\frac{1}{12\lambda\sigma(G_i)+1}-\frac{1}{2}\log(2\pi)\right\}+\gamma_n(\lambda,G_1\ldots,G_n)$$
$$\log \mathbb{P}(L_1=\sigma) \le \sum_{i=1}^{n}\left\{-\lambda[\nu\otimes P(G_i)-\sigma(G_i)]-\lambda\sigma(G_i)\log\frac{\lambda\sigma(G_i)}{\nu\otimes P(G_i)}\right\}$$
$$-\sum_{j=1}^{n}\lambda\left\{\frac{\log[\sigma(G_j)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_j)+\lambda}+\frac{\log(2\pi)}{2\lambda}+\frac{\gamma_n(\lambda,G_1\ldots,G_n)}{\lambda}\right\}$$

Choose $\gamma_2(\lambda)$ as follows:

$$\gamma_2(\lambda)=\frac{\log[\lambda\sigma(G_i)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_i)+\lambda}+\frac{\log(2\pi)}{2\lambda}+\frac{\gamma_n(\lambda,G_1\ldots,G_n)}{\lambda}.$$

We have,

$$\lim_{\lambda\to\infty}\gamma_2(\lambda)=\lim_{\lambda\to\infty}\left[\frac{\log[\lambda\sigma(G_j)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_i)+\lambda}+\frac{\log(2\pi)}{2\lambda}+\frac{\gamma_n(\lambda,G_1\ldots,G_n)}{\lambda}\right]$$
$$=\lim_{\lambda\to\infty}\frac{1}{\lambda}\gamma_n(\lambda,G_1\ldots,G_n),$$

which proves the upper bound.

Further, Let λ be large, we use equation (2.3) to obtain

$$\log \mathbb{P}(L_1=\sigma)\ge\sum_{i=1}^{n}\{-\lambda\nu\otimes P(G_i)+\lambda\sigma(G_i)\log[\lambda\nu\otimes P(G_i)]-\log[\lambda\sigma(G_j)]!\}.$$

Using equation (3.1), we get,

$$\log \mathbb{P}(L_1=\sigma)\ge\sum_{i=1}^{n}\{-\lambda\nu\otimes P(G_i)-\log[(2\pi)^{\frac{1}{2}}(\lambda\sigma(G_i))^{\lambda\sigma(G_i)+\frac{1}{2}}e^{-\lambda\sigma(G_i)}]\}$$
$$+\sum_{i=1}^{n}\left\{\frac{1}{12\lambda\sigma(G_i)}+\lambda\sigma(G_i)\log[\lambda\nu\otimes P(G_i)]\right\}$$
$$\log \mathbb{P}(L_1=\sigma)\ge\sum_{i=1}^{n}\left\{-\lambda\nu\otimes P(G_i)-\frac{1}{2}\log(2\pi)-\left[\lambda\sigma(G_i)+\frac{1}{2}\right]\log[\lambda\sigma(G_i)]+\lambda\sigma(G_j)\right\}$$
$$+\sum_{i=1}^{n}\left\{\frac{1}{12\lambda\sigma(G_i)}+\lambda\sigma(G_i)\log[\lambda\nu\otimes P(G_j)]\right\}$$
$$\log \mathbb{P}(L_1=\sigma)\ge\sum_{i=1}^{n}\left\{-\lambda[\nu\otimes P(G_i)-\sigma(G_j)]-\lambda\sigma(G_i)\log\frac{\sigma(G_i)}{\nu\otimes P(G_i)}-\frac{1}{2}\log[\lambda\sigma(G_j)]\right\}$$
$$+\sum_{i=1}^{n}\left\{\frac{1}{12\lambda\sigma(G_i)}-\frac{1}{2}\log(2\pi)\right\}$$

$$\log\mathbb{P}(L_1=\sigma)\geq\sum_{i=1}^{n}\left\{-\lambda[\nu\otimes P(G_i)-\sigma(G_i)]-\lambda\sigma(G_i)\log\frac{\sigma(G_i)}{\nu\otimes P(G_i)}\right\}$$
$$-\sum_{i=1}^{n}\lambda\left\{\frac{\log[\lambda\sigma(G_i)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_i)}+\frac{\log(2\pi)}{2\lambda}\right\}.$$

Choose $\gamma_1(\lambda)$ as follows:

$$\gamma_1(\lambda)=\frac{\log[\lambda\sigma(G_i)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_i)}+\frac{\log(2\pi)}{2\lambda}$$

We have,

$$\lim_{\lambda\to\infty}\gamma_1(\lambda)=\lim_{\lambda\to\infty}\left[\frac{\log[\lambda\sigma(G_i)]}{2\lambda}-\frac{1}{12\lambda^2\sigma(G_i)}+\frac{\log(2\pi)}{2\lambda}\right]=0$$

which proves the lower bound.

Lemma 3.2: *Let Y be a BRN with rate $\nu_\lambda:\Gamma\to[0,\infty)$ satisfying $\lambda^{-3}\nu_\lambda\to\nu$ and a probability of coverage $P_\lambda:\mathcal{D}\to(0,1)$ that satisfies $\lambda^2P_\lambda\to P$. Then,* $\lim_{\lambda\to\infty}\mathbb{P}(|\ell|\leq 2\lambda)=1$.

Proof: Suppose $G_1, G_2, \cdots, G_m$ is a disjoint decomposition of $\mathbb{R}^d$ such that $\infty>w_k>|\mathcal{Y}\cap G_k|$, where $w_k(G_k):=w_k$ and note $\beta=\max(w_1, w_2, w_3, ..., w_n)$. and $\|\ell_k\|\leq\beta$, for all $k=1, 2, \dots, n$. Take $|\ell|=\sum_{k=1}^{n}|\ell_k|$, where $\ell_k=\mathcal{Y}\cap G_k$ and observe that $|\ell_1|,...,|\ell_n|$ are independent and poisson random variables with parameter $\nu(\mathcal{Y}\cap G_k)$, $k=1, 2, 3, ..., m$, respectively. Therefore, we apple the Bennett's inequality to the random variables $\|\ell_1\|, \dots, \|\ell_n\|$, to obtain

$$\mathbb{P}(|\ell|-\mathbb{E}|\ell|>\lambda)\leq e^{-(\frac{\lambda}{\beta^2}\psi(\beta))},\tag{3.2}$$

while $\psi(a)=(1+a)\log(1+a)-a$. Using (3.2) yields

$$\mathbb{P}(|\ell|\leq\mathbb{E}|\ell|+\lambda)\geq 1-e^{-(\frac{\lambda}{b^2}\phi(b))},$$

which concludes the proof of Lemma (3.2).

The proof of the Theorem 3.3 below follows from the stirling formula. See [9].

Lemma 3.3: *Suppose Y is a BRN with rate $\nu_\lambda:\Gamma\to[0,\infty)$ satisfying $\lambda^{-3}\nu_\lambda\to\nu$ and a coverage probability $P_\lambda:\mathcal{D}\to(0,1)$ that satisfies $\lambda^2P_\lambda\to P$. Then, as $\lambda\to\infty$, L_1 satisfies a large deviation principle with rate function:*

$$I_1(\sigma)=\begin{cases}H(\sigma\|\nu\otimes P), & \text{if } \|\sigma\|=1,\\ \infty, & \text{otherwise.}\end{cases}\tag{3.3}$$

Proof: Write $\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D}) := \{\sigma \in \mathcal{N}(\mathcal{Y}\times\mathcal{D}) : \lambda\sigma(b) \in \mathbb{N} \text{ for all } b \in \mathcal{Y}\}$ and

$$\alpha_n := \max(|\mathcal{X}\times G_1|, |\mathcal{Y}\times G_2|, \ldots, |\mathcal{Y}\times G_n|).$$

Observe that, by construction, we have $|\mathcal{Y}\times G_i| < \infty,$ for all $i = 1, 2, \ldots, n.$ Use Lemma (3.2) and Lemma (3.1) to establish that:

$$\begin{aligned}(1+2\lambda)^{-n\beta_n} e^{-\lambda \inf_{\{\sigma\in G^o\cap\mathcal{N}(\mathcal{Y}\times\mathcal{D})\}} H(\sigma^{(n)}\|\nu^{(n)}\otimes P^{(n)})+\phi_1(\lambda)} &\le \sum_{\sigma\in G^o\cap\mathcal{N}(\mathcal{Y}\times\mathcal{D})} e^{-\lambda H(\sigma^{(n)}\|\nu\otimes P^{(n)})+\phi_2(\lambda)}\\ &\le \mathbb{P}(L_1\in G)\\ &\le \sum_{\sigma\in cl(G)\cap\mathcal{N}(\mathcal{Y}\times\mathcal{D})} e^{-\lambda H(\sigma^{(n)}\|\nu\otimes P^{(n)})+\phi_2(\lambda)}\\ &\le (1+2\lambda)^{n\beta_n} e^{-\lambda T+\phi_2(\lambda)},\end{aligned}$$

where

$$T = \inf_{\sigma\in cl(G)\cap\mathcal{N}(\mathcal{Y}\times\mathcal{D})} H(\sigma^{(n)}\,\|\,\nu^{(n)}\otimes P^{(n)}),$$

$\sigma^{(n)}$ and $\nu^{(n)}\otimes P^{(n)}$ are the coarsening projections of σ and $\nu\otimes Q$ on the decomposition $(G_1,\ldots,G_n)$.

Therefore, we have

$$\begin{aligned}\liminf_{\lambda\to\infty}[-\inf_{\sigma\in G^o\cap\mathcal{N}_\lambda(\mathcal{Y})} H(\sigma^{(n)}\,\|\,\nu\otimes P^{(n)})] &\le \lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}(L_1\in G)\\ &\le \limsup_{\lambda\to\infty}[-\inf_{\sigma\in cl(G)\cap\mathcal{N}_\lambda(\mathcal{Y})} H(\sigma^{(n)}\,\|\,\nu\otimes P^{(n)})].\end{aligned}$$

Observe that $cl(G)\cap\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D})\subset cl(G)$ for every $\lambda\in\mathbb{R}_+$ and therefore we have

$$\limsup_{\lambda\to\infty}[-\inf_{\{\sigma\in cl(G)\cap\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{Y})\}} H(\sigma^{(n)}\,\|\,\nu\otimes P^{(n)})] \le -\inf_{\{\sigma\in cl(G)\}} H(\sigma^{(n)}\,\|\,\nu^{(n)}\otimes P^{(n)}).$$

We apply similar arguments as in [6, page 17], to obtain

$$\liminf_{\lambda\to\infty}[-\inf_{\sigma\in G^o\cap\mathcal{M}_\lambda(\mathcal{Y}\times\mathcal{D})} H(\sigma^{(n)}\,\|\,\nu\otimes P^{(n)})] \ge -\inf_{\sigma\in G^O} H(\sigma^{(n)}\,\|\,\nu^{(n)}\otimes P^{(n)}).$$

Therefore, we have

$$-\inf_{\sigma\in F^0} H(\sigma^{(n)}\,\|\,\nu^{(n)}\otimes P^{(n)}) \le \lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}(L_1\in F) \le -\inf_{\sigma\in cl(F)} H(\sigma^{(n)}\,\|\,\nu^{(n)}\otimes P^{(n)}).$$

Now, when we take $n\to\infty,$ we obtain

$$-\inf_{\sigma\in G^o} H(\sigma\,\|\,\nu\otimes P) \le \lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}(L_1\in G) \le -\inf_{\sigma\in cl(G)} H(\sigma\,\|\,\nu\otimes P),$$

which ends the proof.

4. LDP for L_2 given L_2.

The main method used in this section is the Gartner-Ellis Theorem. Let σ_λ be a sequence of empirical distributions converging to σ.

Lemma 4.1: *Suppose Y is a BRN with rate $\nu_\lambda : \Gamma \to [0,\infty)$ that satisfy $\lambda^{-3}\nu_\lambda \to \nu$ and a coverage law $P_\lambda : \mathcal{D} \to (0,1)$ that satisfy $\lambda^2 P_\lambda \to P$. Assume the connection law $q_\lambda(y, a_y, x, a_x)$ of two devices with their position in $\mathbb{Y}$ satisfy $\lambda q_\lambda(Y, a_Y, X, a_X) \to \nu(a_X - a_Y)P(a_X)P(a_Y)$. Then L_2 conditional on the event $\{L_1 = \sigma_\lambda\}$ satisfies an LDP with rate function given as,*

$$I_\sigma(\mu) = \begin{cases} \frac{1}{2}[H(\mu \| \Psi\sigma \otimes \sigma) + \| \Psi\sigma \otimes \sigma \| - \| \mu \|], & \text{if } \| \mu \| < \infty, \mu_1 = \sigma \\ \infty, & \text{otherwise.} \end{cases} \tag{4.1}$$

Observe from the definition of these two distributions in Section 2, that $\lambda L_2(y, a_y, x, b_x)$ conditional on $L_1(X, b_X) = \sigma_\lambda(X, b_X)$ follows a binomial distribution with parameters, $\lambda^2 \sigma_\lambda(X, b_X)\, \sigma_\lambda\, (Y, a_Y) \,/\, 2$ and $q_\lambda(Y, a_Y, X, a_X)$.

Lemma 4.2: *Suppose Y is a BRN with rate $\nu_\lambda : \Gamma \to [0,\infty)$ satisfying $\lambda^{-3}\nu_\lambda \to \nu$ and a coverage law $P_\lambda : \mathcal{D} \to [0,1]$ that satisfies $\lambda^2 P_\lambda \to P$. Assume the linking probability $q_\lambda(Y, a_Y, X, a_X)$ of two devices with their location in $\mathbb{Y}$ is such that $\lambda q_\lambda(X, a_X, Y, a_Y) \to \nu(a_X - a_Y)\, P(a_X)\, P(a_Y)$. Let L_2 be conditional on the event $\{L_1 = \sigma_\lambda\}$. Then, for any bounded functional, $h : \mathcal{Y} \times \mathcal{D} \times \mathcal{Y} \times \mathcal{D} \to \mathbb{R}$:*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \Phi_\lambda(h) = \Phi(h), \tag{4.2}$$

$$\Phi(h) = -\frac{1}{2} \int_{(X, a_X) \in (\mathcal{Y} \times \mathcal{D})} \int_{(Y, a_Y) \in (\mathcal{Y} \times \mathcal{D})} (1 - e^{h(X, a_X, Y, a_Y)}) \Psi(a_X, a_Y) \sigma_\lambda(dX, da_X) \sigma_\lambda(dY, da_Y).$$

Proof: Let $\Phi_\lambda(h)$ be the m.g.f of L_2 conditional on L_1 Then by definition, we have:

$$\begin{aligned} \Phi_\lambda(h) &= \mathbb{E}[e^{\lambda \langle h, L_2 \rangle} \mid L_1 = \sigma_\lambda] \\ &= \mathbb{E}[e^{\lambda \int\int h(X, a_X, Y, a_Y) L_2(dX, da_X, dY, da_Y)} \mid L_1 = \sigma_\lambda]. \end{aligned}$$

Choose $G_1, \ldots, G_m$ a decomposition of $(\mathcal{Y} \times \mathcal{D})^2$ into subset of locally finite sets and define Φ_λ by

$$\Phi_\lambda(h) = \mathbb{E}\left[\prod_{j=1}^{n} \prod_{(X, a_X, Y, a_Y) \in G_j} e^{\lambda h(X, a_X, Y, a_Y) L_2(dX, da_X, dY, da_Y)} \mid L_1 = \sigma_\lambda \right].$$

As the $G_1,\dots,G_m$ are disjoint of each other, we have:

$$\Phi_\lambda(h)=\prod_{j=1}^{m}\prod_{(X,a_X,Y,a_Y)\in G_j}\mathbb{E}\left[e^{\lambda h(X,a_X,Y,a_Y)L_2(dX,da_X,dY,da_Y)}\,|\,L_1=\sigma_\lambda\right].$$

We note that the probability law of L_2 given $L_1=\sigma_\lambda$ is a binomial distribtion. Thus, we have

$$L_2\,|\,L_1=\sigma_\lambda\sim Bin\left(\frac{\lambda^2}{2}\sigma_\lambda\otimes\sigma_\lambda,q_\lambda\right).$$

We write

$$\Phi_\lambda(h)=\prod_{j=1}^{m}\prod_{(X,a_X,Y,a_Y)\in G_j}[1-q_\lambda(y,a_Y,z,b_z)+q_\lambda(X,a_X,Y,a_Y)e^{h(y,a_Y,z,b_z)}]^{\frac{\lambda^2}{2}\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)}.$$

and observe that we have

$$\log\Phi_\lambda(h)=\Sigma_{j=1}^{m}\int_{(X,a_X,Y,a_Y)\in G_j}\log[1-q_\lambda(X,a_X,Y,a_Y)+q_\lambda(X,a_X,Y,a_Y)e^{h(X,a_X,Y,a_Y)}]^{\frac{\lambda^2}{2}\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)}$$

$$=\Sigma_{j=1}^{m}\int_{(X,a_X,Y,a_Y)\in G_j}\frac{\lambda^2}{2}\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)\log[1-q_\lambda(X,a_X,Y,a_Y)+q_\lambda(X,a_X,Y,a_Y)e^{h(X,a_X,Y,a_Y)}]$$

$$=\frac{1}{2}\Sigma_{j=1}^{m}\int_{(X,a_X,Y,a_Y)\in G_j}\log[1-q_\lambda(X,a_X,Y,a_Y)(1-e^{h(X,a_X,Y,a_Y)})]\lambda^2\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y).$$

Using the Taylor expansion of $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots,$ we get,

$$=\frac{1}{2}\sum_{j=1}^{m}\int_{(X,a_X,Y,a_Y)\in G_j}\left[-q_\lambda(X,a_X,Y,a_Y)(1-e^{g(X,a_X,Y,a_Y)})+O\left(\frac{1}{\lambda^2}\right)\right]\lambda^2\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)$$

if we divide through by λ, we obtain

$$\frac{1}{\lambda}\log\Phi_\lambda(h)=-\sum_{j=1}^{m}\int_{(X,a_X,Y,a_Y)\in G_j}\left[-\lambda q_\lambda(X,a_X,a_Y)(1-e^{h(a_X,a_Y)})+O\left(\frac{1}{\lambda}\right)\right]\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)/2$$

If we allow $\lambda\to\infty,$ and $\sigma_\lambda\to\sigma,$ we obtain:

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\Phi_\lambda(h)=-\frac{1}{2}\sum_{i=1}^{\infty}\int_{(X,a_X,Y,a_Y)\in G_j}(1-e^{h(X,a_X,Y,a_Y)})\Psi(X,a_X,Y,a_Y)\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)$$

$$=-\frac{1}{2}\int_{(\Gamma\times\mathcal{D})^2}(1-e^{h(X,a_X,Y,a_Y)})\Psi(a_X,a_Y)\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)$$

$$=-\frac{1}{2}\int_{(Y,a_Y)\in(\mathcal{Y}\times\mathcal{D})}\int_{(X,a_X)\in(\mathcal{Y}\times\mathcal{D})}(1-e^{h(X,a_X,Y,a_Y)})\Psi(a_X,a_Y)\sigma_\lambda(dX,da_X)\sigma_\lambda(dY,da_Y)$$

$$=-\frac{1}{2}\int_{(Y,a_Y)\in(\mathcal{Y}\times\mathcal{D})}\int_{(X,a_X)\in(\mathcal{Y}\times\mathcal{D})}(1-e^{h(X,a_X,Y,a_Y)})\Psi\sigma_\lambda\otimes\sigma_\lambda(dX,da_X,dY,da_Y)$$

This completes the proof of the Lemma (4.2). The function Φ is differentiable and hence, we use the Gartner-Ellis theorem, to conclude that L_2^0 conditional on $\{L_1=\sigma_\lambda\}$ satisfies an LDP on the scale $\lambda.$ We shall give the the variational formulation of the rate function as follows:

$$I_w(\sigma) = \frac{1}{2}\sup_g\{\langle g,\sigma\rangle - \Phi(g)\} \tag{4.3}$$

The solution to the rate function is given in terms of entropy as:

$$I_\sigma(\mu) = \begin{cases} \frac{1}{2}[H(\mu \| \Psi\sigma\otimes\sigma) + \|\Psi\sigma\otimes\sigma\| - \|\mu\|] & \text{if } \|\mu\| < \infty, \mu_1 = \sigma \\ \infty, & \text{otherwise} \end{cases} \tag{4.4}$$

where,

$$H(\sigma \| \nu) = \sum \sigma \log\frac{\sigma}{\nu}.$$

We observe that rate function is convex as its variational formation, see Equation (4.3), is a the Legendre transform.

5. Joint LDP for the two empirical distributions.

We shall use the method of mixtures would in this Section. We rely on same arguments as the one found in [10] and [11] to proof our main results. To begin we let $\lambda \in (0,\infty)$, and we write

$$\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D}) := \{\nu \in \mathcal{N}(\mathcal{Y}\times\mathcal{D}) : \lambda\nu(x,a_x) \in \mathbb{N},\ \text{for all}\ (x,a_x) \in \mathcal{Y}\times\mathcal{D}\},$$

$$\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D}\times\mathcal{Y}\times\mathcal{D}) := \{\sigma \in \mathcal{N}(\mathcal{Y}\times\mathcal{D}\times\mathcal{Y}\times\mathcal{D}) : \lambda\sigma(x,b_x,y,a_y) \in \mathbb{N},$$
$$\text{for all}\ (x,b_x,y,a_y) \in \mathcal{Y}\times\mathcal{D}\times\mathcal{Y}\times\mathcal{D}\}.$$

Denote by $\Theta_\lambda := \mathcal{N}_\lambda(\mathcal{Y})$ and $\Theta := \mathcal{N}(\mathcal{Y})$. Define the following probability distribution:

$$Q^{(\lambda)}_{\sigma_\lambda}(\sigma_\lambda) := \mathbb{P}(L_2 = \sigma_\lambda \mid L_1 = \sigma_\lambda)$$

and

$$Q^{(\lambda)}(\sigma_\lambda) := \mathbb{P}(L_1 = \sigma_\lambda),$$

We shall observe that the joint Law of L_1 and L_2 is a mixture of $Q^{(\lambda)}_{\sigma_\lambda}$ and $Q^{(\lambda)}(\sigma_\lambda) := \mathbb{P}(L_1 = \sigma_\lambda)$ and given by

$$d\tilde{P}^\lambda(\sigma_\lambda,\sigma_\lambda) := dQ^{(\lambda)}_{\sigma_\lambda}(\sigma_\lambda)dQ^{(\lambda)}(\sigma_\lambda).$$

Biggins [3] gives the condition under which LDPs for mixtures exists and how to check the goodness of the rate function of the rate function given that the individual LDPs exists. In the Lemmas we check the validity of the LDPs for L_1, L_2 (mixed together) and the goodness of the rate function for the joint LDP for (L_1, L_2). Observe that the sequence of probability distributions $(P^\lambda : \lambda \in (0,\infty))$ is exponentially tight onthe space Θ. Denote by $|E|$ be the cardinality of the edge set E

Lemma 5.1: *The sequence of probability distributions* $(\tilde{P} : \lambda \in \mathbb{R}_+)$ *is exponentially tight onthe space* $\Theta \times \mathcal{N}(\mathcal{Y}\times\mathcal{D}\times\mathcal{Y}\times\mathcal{D})$.

Proof: Let $t := 1-(1-e^l)e^{-\eta}, \eta > 0$ and use the Chebyshev's inequality and Lemma (4.2),to obtain, for sufficiently large λ,

$$\begin{aligned}\mathbb{P}(|E| \le \lambda^2 l) &\le e^{-\lambda^2 l}\mathbb{E}(\exp(|E|)) \\ &\le e^{-\lambda^2 l}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}e^k(e^{-\eta})^k(1-e^{-\eta})^{j-k}\frac{e^{-\lambda}\lambda^j}{j!} \\ &\le e^{-\lambda^2 l}e^{-\lambda}e^{t\lambda}.\end{aligned}$$

Let $u \in \mathbb{N}$, choose $q < u$. We note that for sufficiently large λ, we have that

$$\mathbb{P}(|E| \le \lambda^2 u) \le e^{-\lambda^2 u}.$$

Hence,

$$\mathbb{P}[|E| \le \lambda^2 s/2] \le e^{-\lambda^2 s/2},$$

and this concludes the proof of Lemma (5.1).

Lemma 5.2: I_σ *is a lower semi-continuous function.*

Proof: Note that I_σ is a function of relative entropy, a lower semi-continuous function, plus a linear function. Therefore I_σ is lower semi-continuous function.

We use [3, Theorem 5(b)], Lemma 5.1, Lemma 5.2 and the two LDPs Lemma 3.3 and Lemma 4.1 to establish Theorem 2.3. These results gurantee that under $(\tilde{P}^\lambda)$ the random variables (L_1, L_2) obeys an LDP on the space $\mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D}) \times \mathcal{N}_\lambda(\mathcal{Y}\times\mathcal{D}\times\mathcal{Y}\times\mathcal{D})$ with good rate function I which concludes the proof of Theorem 2.3.

6. Conclusion

In this article we have determined a joint large deviation principle for the empirical pairdistribution and the empirical markeddistribution in a telecommunication system modelled as the Boolean random network. The large deviation principles developed in this article could form the bases of solving an optimization problem that involves the Gibbs distribution on a spatial telecommunication system modelled as the Boolean random network.

Acknowledgement

This article forms part of the MSc. work research of the first author presented to AIMS, Ghana. It has has already appeared as a preprint on the Arxiv archives. See, Boahen et.al. [4].

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] F. Baccelli and B. Blaszczyszyn, Stochastic Geometry and Wireless Networks, Vol. I, Theory, Now Publishers Inc. (2009).

[2] F. Baccelli and B. Blaszczyszyn, Stochastic Geometry and Wireless Networks, Vol. II, Applications, Now Publishers Inc. (2009).

[3] J. D. Biggins, "Large deviations for mixtures," *Electron. Comm. Probab.*, vol. 9, pp. 60–71 (2004).

[4] A. K. Boahen, T. Katsekpor, and K. Doku-Amponsah, "Large deviations for spatial telecommunication systems: The boolean model," arXiv (2021). [Online]. Available: https://arxiv.org/pdf/2108.11820.pdf.

[5] K. Doku-Amponsah, "Large deviations and basic information theory for hierarchical and networked data structures," Ph.D. Thesis, University of Bath, Bath, U.K., ProQuest Dissertations Publishing (2006).

[6] A. Dembo and O. Zeitouni, Large Deviations: Techniques and Applications, 2nd ed., Springer, New York (1998).

[7] M. Haenggi, Advanced Topics in Random Wireless Networks, Course notes (2010).

[8] C. M. Hirsch, B. Jahnel, P. Keeler, and R. Patterson, "Large deviation principles for connectable receivers in wireless networks," *Advances in Applied Probability,* vol. 48, no. 4, pp. 1061–1094 (2016).

[9] H. Robins, "A Remark on Stirling's Formula," *The American Mathematical Monthly*, vol. 62, no. 1, pp. 26–29 (1995).

[10] E. Sakyi-Yeboah, L. Asiedu, and K. Doku-Amponsah, "Local large deviation principle, large deviation principle, and information theory for the signal-to-interference-plus-noise ratio graph models," *Journal of Information and Optimization Sciences,* vol. 42, no. 1, pp. 249–273 (2020).

[11] E. Sakyi-Yeboah, P. Andam, L. Asiedu, and K. Doku-Amponsah, "Large deviations, Sharron-MacMillan-Breiman Theorem for supercritical telecommunication networks," *Journal of Information and Optimization Sciences,* vol. 42, no. 7, pp. 1665–1683 (2020).

[12] A. T. Matthew and A. Kottas, "Mixture Modeling for Marked Poisson Processes," *Bayesian Analysis*, vol. 7, no. 2, pp. 335–362 (2012).

[13] I. Umar, A. Lotis, and K. Doku-Amponsah, "From Statistical Mechanics to Large Deviations of Uniformly Random D-Regular Graphs," *Journal of Discrete Mathematics and Cryptography,* vol. 24, no. 6, pp. 1767–1773 (2021).

Received June, 2022

Revised October, 2022